新迭代法的构造方法及应用

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摘 要:介绍并讨论了利用两个辅助函数 $z=g(x)$ 、$u(x)=f(x)e^{\alpha x}$ 和差商来构造迭代法的几种方法。经过选择适当的辅助函数及差商,构造了以前几种常用的迭代方法,最后构造了一种新的迭代法即对数迭代法,此迭代法包含两个参数,具有很强的适应能力。

关键词:非线性方程;迭代法;二阶收敛; 差商
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考虑数值求解非线性方程

$$f(x)=0 \tag{1}$$

其中,实值函数 $f(x)$ 在实零点 x^* 的某邻域 $U(x^*)$ 内连续可微,且 $f'(x)\neq 0$ 。

牛顿法是科学与工程计算中数值求解式(1)的常用数值方法,它一般至少是二阶收敛的,但它需要调用导数值以及没有调节收敛速度的参数,使其应用受到一定限制,从而相应出现了二阶收敛指数迭代法[1]和带有参数的牛顿迭代法和指数迭代法[2]及具有参数的不带有导数的牛顿迭代法和指数迭代法[3],这些方法都是借用动力系统通过不同求解方法以及用差商代替导数得到的不同迭代法。

本文从数学角度通过辅助函数构造新的迭代法,并给出敛速估计,然后给出了上述迭代法的辅助函数,并利用新的辅助函数给出了一类新的迭代法。

1 新迭代法的构造

首先,引入辅助函数 $z=g(x)$,$g(x)$ 在实零点 x^* 某邻域 $U(x^*)$ 内连续可微,且 $g'(x)\neq 0$,故在 $U(x^*)$ 内存在反函数 $x=g^{-1}(z)$,则式(1)求解变换为求非线性方程

$$h(z)=f(g^{-1}(z))=f(x)=0 \tag{2}$$

的实零点 z^* 满足 $h(z^*)$,又 $h'(z^*)=f'(x^*)[g^{-1}(z^*)]'=f'(x^*)/g'(x^*)\neq 0$ 由牛顿迭代法公式得

$$z_{n+1}=z_n-\frac{h(z_n)}{h'(z_n)}$$

并得迭代公式

$$g(x_{n+1})=g(x_n)-\frac{g'(x_n)f(x_n)}{f'(x_n)} \tag{3}$$

其次,为得到参数引入辅助函数 $u(x)=f(x)e^{\alpha x}(\alpha\in\mathbf{R})$,可知函数 $u(x)$ 与函数 $f(x)$ 同解,利用式(2)变换方法,令

$w(z)=u(g^{-1}(z))$,$u(x^*)=0$,$u'(x^*)=\alpha e^{\alpha x}f(x^*)+e^{\alpha x^*}f'(x^*)\neq 0$, $w'(z^*)=u'(x^*)/g'(x^*)\neq 0$ 得

$$z_{n+1}=z_n-\frac{w(z_n)}{w'(z_n)}$$

$$g(x_{n+1})=g(x_n)-\frac{g'(x_n)u(x_n)}{u'(x_n)}$$

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$$g(x_{n+1})=g(x_n)-\frac{g'(x_n)f(x_n)}{\alpha f(x_n)+f'(x_n)} \tag{4}$$

再次,为得到不含导数值的迭代式,在式(4)中用差商$(f(x_n+f(x_n))-f(x_n))/f(x_n)$代替导数$f'(x_n)$便得到不带导数带有参数的迭代法

$$g(x_{n+1})=g(x_n)-\frac{g'(x_n)f^2(x_n)}{\alpha f^2(x_n)+f(x_n+f(x_n))-f(x_n)} \tag{5}$$

2 二阶收敛性和敛速估计

仅在当$f''(x^*)$存在时,得到如下二阶收敛性和敛速估计。

定理 1 设函数$f(x)$满足$f(x^*)=0$,$f'(x^*)\neq 0$且$f''(x^*)$存在,函数$z=g(x)$满足$g(x^*)\neq 0$,且$g''(x^*)$存在,则当$x_0\in U(x^*)$时,迭代公式(3)所得序列$\{x_n\}$至少是二阶收敛的,并有

$$r(x^*)=\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}(\text{其中 } e_n=x_n-x^*) \tag{6}$$

证明 设x_n在x^*的去心邻域$U^0(x^*)$中,由带 Peano 余项的 Talor 公式得

$$f(x_n)=f'(x^*)e_n+f''(x^*)e_n^2/2+o(e_n^2)$$
$$f'(x_n)=f'(x^*)+f''(x^*)e^2+o(e_n^2)$$
$$g(x_n)=g'(x^*)e_n+g''(x^*)e_n^2/2+o(e_n^2)$$
$$g'(x_n)=gf'(x^*)+g''(x^*)e^2+o(e_n^2)$$

故

$$g'(x_n)e_{n+1}+o(e_{n+1})=g'(x^*)e_n+g''(x^*)e_n^2/2+o(e_n^2)-$$
$$\frac{(g'(x^*)+g''(x^*)e_n+o(e_n))(f'(x^*)e_n+f''(x^*)e_n^2/2+o(e_n^2))}{f'(x^*)+f''(x^*)e_n+o(e_n^2)}=$$
$$\frac{\frac{1}{2}g'(x^*)+f''(x^*)e_n{}^2-\frac{1}{2}+g''(x^*)f'(x^*)e_n^2/2+o(e_n^2))}{f'(x^*)+f''(x^*)e_n+o(e_n^2)}$$

两边同除$g'(x^*)e_n^2$得

$$\frac{e_{n+1}}{e_n^2}(1+o(1))=\frac{\frac{1}{2}+f''(x^*)-g''(x^*)f'(x^*)e_n^2/2\,g'(x^*)+o(1)}{f'(x^*)+f''(x^*)e_n+o(e_n)}$$

两边取极限得

$$r(x^*)=\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}$$

定理 2 设函数$f(x)$满足$f(x^*)=0$,$f'(x^*)\neq 0$且$f''(x^*)$存在,函数$z=g(x)$满足$g(x^*)\neq 0$,且$g''(x^*)$存在,则当$x_0\in U(x^*)$时,迭代式(4)所得序列$\{x_n\}$至少是二阶收敛的,并有

$$r(x^*)=\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}+\alpha(\text{其中 } e_n=x_n-x^*) \tag{7}$$

证明 由定理 1 得

$$r_\alpha(x^*)=\frac{u''(x^*)}{2u'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}=\frac{\alpha^2e^{\alpha x^*}f(x^*)+2\alpha e^{\alpha x^*}f'(x^*)+e^{\alpha x^*}f''(x^*)}{\alpha e^{\alpha x^*}f(x^*)+e^{\alpha x^*}f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}=$$
$$\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}+\alpha$$

定理 3 设函数$f(x)$满足$f(x^*)=0$,$f'(x^*)\neq 0$且$f''(x^*)$存在,函数$z=g(x)$满足$g(x^*)\neq 0$,且$g''(x^*)$存在,则当$x_0\in U(x^*)$时,迭代式(5)所得序列$\{x_n\}$至少是二阶收敛的,并有

$$r_\alpha(x^*)=\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}+\frac{f''(x^*)}{2}+\alpha \tag{8}$$

证明 由带 Peano 余项的 Talor 公式得

$$f(x_n+f(x_n))-f(x_n)=f'^2(x^*)e_n+(2+f'(x^*))f'(x^*)f''(x^*)e_n^2/2+o(e_n^2)$$

$$(f(x_n+f(x_n))-f(x_n))/f(x_n)=\frac{f'^2(x^*)+(3+f'(x^*))f'(x^*)f''(x^*)e_n/2}{f'(x^*)e_n+f''(x^*)e_n^2/2}$$

$$g'(x^*)e_{n+1}+o(e_{n+1})=g'(x^*)e_n+g''(x^*)e_n^2/2+o(e_n^2)-$$

$$\frac{(g'(x^*)+g''(x^*)e_n+o(e_n))(f'(x^*)e_n+f''(x^*)e_n^2/2+o(e_n^2))}{\alpha f'(x^*)e_n+\dfrac{f'^2(x^*)+(3+f'(x^*))f'(x^*)f''(x^*)e_n/2}{f'(x^*)+f''(x^*)e_n/2}+o(e_n)}$$

$$g'(x^*)e_{n+1}+o(e_{n+1})=\frac{\frac{1}{2}+f''(x^*)g'(x^*)f'(x^*)-g''(x^*)f'^2(x^*)+f''(x^*)g'(x^*)f'^2(x^*))e_n^2+o(e_n^2)}{f'^2(x^*)+(3+f'(x^*))f'(x^*)f''(x^*)E_N/2+\alpha(f'(x^*)+f''(x^*)e_n/2)f'(x^*)e_n}$$

两边同除 $g'(x^*)e_n^2$取极限得

$$r_\alpha(x^*)=\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}+\frac{f''(x^*)}{2}+\alpha$$

3　构造公式的应用

在式(4)和式(7)中令 $g(x)=x$ 得带参数的牛顿迭代法公式及敛速估计式

$$x_{n+1}=x_n-\frac{f(x_n)}{\alpha f(x_n)+f'(x_n)} \tag{9}$$

$$r(x^*)=\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=\frac{f''(x^*)}{2f'(x^*)}-\frac{g''(x^*)}{2g'(x^*)}+\alpha \tag{10}$$

令 $\alpha=o$ 即为牛顿迭代法公式及敛速估计式。

在式(4)和式(7)中令 $z=g(x)=\ln x(x\in\mathbf{R}^+)$ 得迭代法公式及敛速估计式

$$\ln x_{n+1}=\ln x_n-\frac{f(x_n)}{x_n[\alpha f(x_n)+f'(x_n)]}\quad(x_n>0(n=0,1,2,\cdots))$$

$$x_{n+1}=x_n\exp\{-f(x_n)/[x_n(\alpha f(x_n)+f'(x_n))]\}\quad(x_n>0(n=0,1,2,\cdots)) \tag{11}$$

$$r(x^*)=\frac{f''(x^*)}{2f'(x^*)}+\frac{1}{2x^*}+\alpha \tag{12}$$

在式(4)和式(7)中令 $z=g(x)=\ln x(x\in\mathbf{R}^-)$ 得迭代法公式及敛速估计式

$$\ln(-x_{n+1})=\ln(-x_n)-\frac{f(x_n)}{x_n[\alpha f(x_n)+f'(x_n)]}\quad(x_n<0(n=0,1,2,\cdots))$$

$$x_{n+1}=x_n\exp\{-f(x_n)/[x_n(\alpha f(x_n)+f'(x_n))]\}\quad(x_n<0(n=0,1,2,\cdots)) \tag{13}$$

$$r(x^*)=\frac{f''(x^*)}{2f'(x^*)}+\frac{1}{2x^*}+\alpha \tag{14}$$

由式(11)~(14)合并得带参数指数迭代法公式及敛速估计式

$$x_{n+1}=x_n\exp\{-f(x_n)/[x_n(\alpha f(x_n)+f'(x_n))]\}\quad(x_n\neq 0(n=0,1,2,\cdots)) \tag{15}$$

$$r(x^*)=\frac{f''(x^*)}{2f'(x^*)}+\frac{1}{2x^*}+\alpha \tag{16}$$

令 $\alpha=0$,即为指数迭代法公式及敛速估计式。

在式(9)中用差商$(f(x_n+f(x_n))-f(x_n))/f(x_n)$ 代替导数$f'(x_n)$,由式(5)和式(8)便得到不带导数带有参数的牛顿迭代法及敛速估计式

$$x_{n+1}=x_n-\frac{f^2(x_n)}{\alpha f^2(x_n)+f(x_n+f(x_n))-f(x_n)}$$

$$r_\alpha(x^*)=\frac{f''(x^*)}{2f'(x^*)}+\frac{f''(x^*)}{2}+\alpha$$

在式(15)中用差商$(f(x_n+f(x_n))-f(x_n))/f(x_n)$ 代替导数$f'(x_n)$,由式(5)和式(8)便得到不带导数带有参数的指数迭代法及敛速估计式

$$x_{n+1}=x_n\exp\{-f^2(x_n)/[x_n(\alpha f^2(x_n)+f(x_n+f(x_n))-f(x_n))]\}$$

$$r_\alpha(x^*)=\frac{f''(x^*)}{2f'(x^*)}+\frac{f''(x^*)}{2}+\frac{1}{2x^*}+\alpha$$

现在构造一新的迭代法,称之为双参数对数迭代法,令 $g(x)=e^{\beta x}$,由式(4)和式(7)得

$$e^{\beta x_{n+1}}=e^{\beta x_n}-\frac{\beta e^{\beta x_n}f(x_n)}{\alpha f(x_n)+f'(x_n)}$$

$$x_{n-1}-x_n=\frac{1}{\beta}\ln(1-\frac{\beta f(x_n)}{\alpha f(x_n)+f'(x_n)}) \tag{17}$$

$$r(x^*)=\frac{f''(x^*)}{2f'(x^*)}-\frac{\beta}{2}+\alpha$$

在式(17)中用差商 $(f(x_n+f(x_n))-f(x_n))/f(x_n)$ 代替导数 $f'(x_n)$,由式(5)和式(8)便得不带导数形式的迭代法及敛速估计式

$$x_{n-1}-x_n=\frac{1}{\beta}\ln(1-\frac{\beta f^2(x_n)}{\alpha f^2(x_n)+f(x_n+f(x_n))-f(x_n)})$$

$$r_\alpha(x^*)=\frac{f''(x^*)}{2f'(x^*)}+\frac{f''(x^*)}{2}-\frac{\beta}{2}+\alpha$$

可以看出双参数对数迭代法有两个不同的参数,显然增强了一定的适应能力,当然也有一定不足之处,它必须满足对数函数内部分始终大于零。

从数学角度通过两个辅助函数及差商来构造新迭代法,很容易看清迭代法的来龙去脉,迭代法通式及敛速估计通式更便于统一分析,而且根据敛速估计可以选择相应的辅助函数及相应参数值来选择迭代法以满足需要。

参考文献:

[1] 吴新元. 解非线性方程的二阶敛速指数迭代法[J]. 计算数学,1998,20(4):367-370.

[2] 吴新元. 对牛顿迭代法的一个重要修改[J]. 应用数学和力学,1999,20(8):863-866.

[3] 郑 权. 具有参数的不带有参数的平方收敛的迭代法[J]. 计算数学,2003,25(1):107-112.

[4] 高红霓,曹泽阳. 一种新的非线性方程求根迭代法[J]. 空军工程大学学报(自然科学版),2002,3(2):84-86.

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Construction Methods of New Iterative Method and Their Applications

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Abstract: Some methods of constructing a new iterative method by means of two auxiliary functions, z = g (x) and u (x) = f (x) eax , and difference quotient are introduced and discussed. Through selecting and adopting proper auxiliary functions and difference quotient in constructing some preceding iterative methods, a new logarithm iterative method is deduced and constructed. This method contains two parameters and is very good in adoptability.

Abstract: nonlinear equation; iterative method; quadratic convergence; difference quotient